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Technical Comment

Maximizing the Determinant of the Information Matrix with the Effective Independence Method

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A METHOD has been presented by Kammer[1] that addresses the problem of optimally placing sensors on a large space structure for the purpose of on-orbit modal testing. The method ranks potential sensor sites according to their contribution to the independence of the target modes and iteratively deletes the sites that have the lowest ranking. It is implied[1] that deleting sensor sites in this way tends to maximize the determinant of the Fisher information matrix (FIM). Numerical examples are given to demonstrate that the method maintains a larger determinant for the FIM when compared with the kinetic energy method of determining optimal sensor locations. However, no relationship between the effective independence ranking and the determinant of the FIM was proven.

The purpose of this comment is to provide a proof that deleting the potential sensor location with the smallest effective independence distribution (E_D) value will produce the smallest relative change in the determinant of the information matrix, and so this method does provide a local maximization of the determinant of the FIM.[2] The formal statement is included in the following theorem where the notation closely follows that of the original paper.

Theorem: Let A and B, respectively, be the Fisher information matrices before and after the ith sensor is deleted; then $\det B = (1 - E_{Di}) \det A$, where E_{Di} is the effective independence distribution of the ith sensor site.

The theorem is a direct consequence of the following lemma.

Lemma: If $B = A - R_i R_i^T$, where R_i is a column vector, then

$$\det B = \det A - R_i^T (A^{\text{cof}})^T R_i$$

with the ith column of A^{cof} containing the cofactors of the ith row of A.[3]

Proof: Since $B = A - R_i R_i^T$, then

$$\det B = \det (A - R_i R_i^T) = \det \begin{bmatrix} a_{11} - r_1 r_1 & \cdots & a_{1k} - r_1 r_k \\ \vdots & \vdots & \vdots \\ a_{k1} - r_k r_1 & \cdots & a_{kk} - r_k r_k \end{bmatrix} \tag{1}$$

where a_{ij} is the ijth term of the k square matrix A and r_j is the jth element of the column vector R_i. Because the determinant is linearly dependent on a single row,[3] Eq. (1) becomes

$$\det B = \det \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ a_{21} - r_2 r_1 & \cdots & a_{2k} - r_2 r_k \\ \vdots & \vdots & \vdots \\ a_{k1} - r_k r_1 & \cdots & a_{kk} - r_k r_k \end{bmatrix} - \det \begin{bmatrix} r_1 r_1 & \cdots & r_1 r_k \\ a_{21} - r_2 r_1 & \cdots & a_{2k} - r_2 r_k \\ \vdots & \vdots & \vdots \\ a_{k1} - r_k r_1 & \cdots & a_{kk} - r_k r_k \end{bmatrix} \tag{2}$$

Expanding both of the matrices in Eq. (2) in the same manner for the second row yields four determinants. One of these determinants is zero, because the first two rows are linearly dependent, thus leaving three nonzero determinants. Continuing this row by row expansion of the determinants yields the following sum:

$$\det B = \det \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} - \det \begin{bmatrix} r_1 r_1 & \cdots & r_1 r_k \\ \vdots & \vdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \cdots - \det \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \vdots & \vdots \\ r_k r_1 & \cdots & r_k r_k \end{bmatrix} \tag{3}$$

Using the definition of the determinant as the sum of row j times the cofactors of the jth row, we can write Eq. (3) as

$$\det B = \det A - r_1 \sum_{j=1}^{k} r_j A_{1j}^{\text{cof}} - \cdots - r_k \sum_{j=1}^{k} r_j A_{kj}^{\text{cof}} \tag{4}$$

In matrix form, Eq. (4) is

$$\det B = \det A - R_i^T (A^{\text{cof}})^T R_i \tag{5}$$

and the lemma is proved. It is now possible to prove the theorem.

Proof: In the context of the effective independence distribution method, A and A^{cof} are symmetric, so

$$\det B = \det A - R_i^T (A^{\text{cof}}) R_i = \det A (1 - R_i^T A^{-1} R_i)$$

where the second equality follows from the relation between the cofactor matrix and the inverse. Since by definition $E_{Di} = R_i A^{-1} R_i$,[1] the theorem is proved. □

From the theorem, it is also seen that $0 \le E_{Di} \le 1$, a result also obtained by Kammer by a different approach. If a sensor location is deleted when $E_{Di} = 1$, then $\det B = 0$, and some linear combination of the target modes will not be recovered. If $E_{Di} = 0$, then $\det B = \det A$, and there is no loss of information when the sensor is deleted. It should be noted that since

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this is a greedy algorithm for a discrete optimization problem, there is no guarantee that the global maximum will be reached. However, the theorem demonstrates that the effective independence distribution is an exact measure of the information that will be lost when deleting one potential sensor location, and in practical applications the method has been shown[1,2] to be effective for optimally locating sensors.

References

[1]Kammer, D. C., "Sensor Placement for On-Orbit Modal Identification and Correlation of Large Space Structures," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 2, 1991, pp. 251-259.

[2]Poston, W. L., "Optimal Sensor Locations for On-Orbit Modal Identification of Large Space Structures," M.S. Thesis, Dept. of Civil, Mechanical, and Environmental Engineering, George Washington Univ., Hampton, VA, July 1991.

[3]Strang, G., *Linear Algebra and Its Applications,* 3rd ed., Harcourt Brace Jovanovich, San Diego, CA, 1988, pp. 214-234.

Reply by Author to W. L. Poston and R. H. Tolson

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THE author would like to thank Poston and Tolson for their interest in the effective independence method of sensor placement he presented in Ref. 1. It was believed that the effective independence method ranked candidate sensor locations such that the deletion of the lowest ranked sensor resulted in the smallest change in the determinant of the Fisher information matrix (FIM). However, this hypothesis had not been proven prior to Poston and Tolson's comment. Their result is elegant and its proof convincing; however, the author would like to present an alternate proof recently brought to his attention by L. Yao, a graduate student in the Department of Electrical and Computer Engineering at the University of Wisconsin. The proof starts with a lemma from Ref. 2.

Lemma: Let $C \in R^{n\times m}$, $D \in R^{m\times n}$, and I_p be a $p \times p$ identity matrix. Then

$$\det(I_n - CD) = \det(I_m - DC) \tag{1}$$

Proof: See the Appendix in Ref. 2. □

The following theorem states Poston and Tolson's result in a slightly different way:

Theorem: $\forall A = \Phi^T\Phi \in R^{n\times n}$ and A positive definite. Let $r_i \in R^{1\times n}$ be the ith row vector of Φ and $B = A - r_i^T r_i$. Then

$$\det(B) = \det(A)(1 - E_{Di}) \tag{2}$$

where $0 \le E_{Di} \le 1$.

Note that $r_i = R_i^T$, which is used in both the comment and Ref. 1.

Proof:

$$\det(B) = \det(A - r_i^T r_i) = \det\left[A(I - A^{-1} r_i^T r_i)\right] \tag{3}$$

Since A and $(I - A^{-1} r_i^T r_i)$ are both square matrices,

$$\det(B) = \det(A)\det(I - A^{-1} r_i^T r_i) = \det(A)\det(1 - r_i A^{-1} r_i^T) = \det(A)(1 - E_{Di}) \tag{4}$$

where the foregoing lemma has been used and $E_{Di} = r_i A^{-1} r_i^T$. Recall that

$$B = A - r_i^T r_i = \sum_{\substack{j=1 \\ j\ne i}}^{n} r_j^T r_j = \Gamma_i^T \Gamma_i \tag{5}$$

where $\Gamma_i = [r_1^T, \ldots, r_{i-1}^T, r_{i+1}^T, \ldots, r_n^T]^T$. Since matrix B can be expressed in this factored form, it must be positive semidefinite. This implies that $\det(B) \ge 0$ and thus $E_{Di} \le 1$. Because A is assumed to be positive definite, A^{-1} is also positive definite. Therefore, $\forall r_i \ne 0$, $i = 1, \ldots, n$,

$$E_{Di} = r_i A^{-1} r_i^T > 0 \tag{6}$$

However, r_i could be a zero row in Φ; therefore, $E_{Di} \ge 0$. This completes the proof. □

Thus, the effective independence sensor placement method iteratively deletes candidate sensor locations that have the smallest impact on the value of the Fisher information matrix determinant.

References

[1]Kammer, D. C., "Sensor Placement for On-Orbit Modal Identification and Correlation of Large Space Structures," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 2, 1991, pp. 251-259.

[2]Kailath, T., *Linear Systems,* Prentice-Hall, Englewood Cliffs, NJ, 1980.

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Errata

Minimizing Selective Availability Error on Satellite and Ground Global Positioning System Measurements

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[J. Guidance 15(5), pp. 1306–1309 (1992)]

BEGINNING with the first full paragraph on page 1308 and continuing on page 1309, six paragraphs appear out of order. The AIAA Editorial Staff regrets this error and any inconvenience it has caused our readers. The correct order appears below:

Since carrier-phase data noise is intrinsically low, smoothing over the entire 5-min integration time is not necessary. Instead of removing the satellite dynamics with a good model, a low-order polynomial interpolation over a short time period can be used for the compression of carrier phase. The simulation analysis in the following section indicates that for Topex data at 1-s intervals, a cubic interpolation over four points every 5 min is appropriate even with the strong Topex dynamics. For ground data, a compression scheme with a quadratic